

Chaos Driven by Soft-Hard Mode Coupling in Thermal Yang-Mills Theory

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Abstract

We argue on a basis of a simple few mode model of $SU(2)$ Yang-Mills theory that the color off-diagonal coupling of the soft plasmon to hard thermal excitations of the gauge field drives the collective plasma oscillations into chaotic motion despite the presence of the plasmon mass.

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The chaoticity of classical nonabelian gauge fields was originally discovered for spatially homogeneous gauge potentials [1] and later extended to special, radially symmetric field configurations [2,3]. More recently, the chaotic dynamics of the full classical Yang-Mills equations was investigated in numerical studies of Hamiltonian $SU(N_c)$ lattice gauge theory with $N_c = 2, 3$ color degrees of freedom [4,5]. The complete Lyapunov spectrum of the $SU(2)$ gauge field was calculated [6], and it was found that the field configurations ergodically cover the available phase space on the time scale of the inverse of the leading Lyapunov exponent, $\tau \approx 1/\lambda_0 \propto 1/g^2 E$, where E is the energy per lattice degree of freedom. The leading Lyapunov exponent coincides with twice the plasmon damping rate obtained in resummed perturbation theory at finite temperature T [7].

The coupling to the heat bath gives rise to a mean inertia of the plasmon field. For long wavelength ($k \rightarrow 0$) and static fields ($\omega = 0$) it coincides with the Debye mass square $m_P^2 = \frac{N_c}{3} g^2 T^2$; for on-shell plasmons in the limit $k \rightarrow 0$ one has $m_P^2 = \frac{N_c}{9} g^2 T^2$. Such a mass restricts the region of chaotic motion to large amplitude plasmon fields. In fact a control parameter,

$$R = \frac{m_P^4 V}{16g^2 E} \quad (1)$$

with E being the total classical energy of the plasmon field and V the plasma volume, has been found to control chaos in simple, few-mode models of the Yang-Mills field theory [8]. For small $R \leq 0.2$ chaos develops. If the mean plasmon mass m_P were the sole effect of hard thermal gluons, then small amplitude plasmon fields would not grow chaotic [9].

We now argue that even small amplitude collective plasma oscillations can be chaotic because of the color-nondiagonal coupling to the fluctuating components of the plasmon self-energy. These color non-diagonal matrix elements can render some eigenvalues of the mass square matrix M_P negative. This is a source of instability for small amplitude soft fields. Once the energy stored in the soft components has grown, the classical chaotic behavior sets in.

We are not able yet to prove the above scenario generally. In this paper we study as an

example the coupled dynamics of some selected Fourier components of the vector potential to show how the mechanism discussed above can be realized. The central point here is the existence of a hierarchy of relevant timescales in the dynamics of oscillations of the gauge field. In fact, three such timescales can be identified.¹ The hard thermal modes oscillate over a period of $t_1 \sim 1/T$. Due to the nonabelian coupling they influence the soft mode dynamics on the timescale $t_2 \sim 1/gT$. This also sets the timescale for the evolution of the soft plasma oscillations. Finally, the thermal ensemble average over particular hard mode configurations is characterized by the timescale of the gluon damping rate $t_3 \sim 1/g^2T$. The fact that $t_2 \leq t_3$ requires that the ensemble average has to be taken over solutions of the plasmon equation of motion rather than over the equation itself.

Let us now consider a simplified model retaining the essential features. Here we build on the infrared limit of Yang-Mills field theory (“Yang-Mills mechanics” [1]). Each Fourier mode of the $SU(2)$ gauge field representing hard thermal gluons carries six degrees of freedom (two polarizations, three internal degrees of freedom). Soft modes, neglecting spatial derivatives, can be described by the three eigenvalues of the $O(3) \times O(3)$ —symmetric vector potential A_i^a . Without loss of generality they can be chosen as [10]:

$$x = A_1^1, \quad y = A_2^2, \quad z = A_3^3. \quad (2)$$

The most general ansatz describing one hard and one soft momentum contains therefore nine coupled modes

$$\begin{aligned} A_1^1 &= x + w_1^1, & A_2^1 &= w_2^1, & A_3^1 &= 0, \\ A_1^2 &= w_1^2, & A_2^2 &= y + w_2^2, & A_3^2 &= 0, \\ A_1^3 &= w_1^3, & A_2^3 &= w_2^3, & A_3^3 &= z. \end{aligned} \quad (3)$$

Here

¹We here assume, as is customary in the analysis of thermal gauge theories, that g is sufficiently small that the hierarchy $t_1 \ll t_2 \ll t_3$ holds.

$$w_i^a = \sqrt{2} \left(u_i^a \cos(\vec{k} \cdot \vec{r}) + v_i^a \sin(\vec{k} \cdot \vec{r}) \right) \quad (4)$$

denote the hard thermal modes with time-dependent amplitudes u_i^a and v_i^a . We choose the thermal wave number $\vec{k} = T\vec{e}_3$ to point in the third direction. Then all longitudinal components $w_3^a = 0$ vanish and only one derivative (in the third direction) differs from zero in this ansatz:

$$\partial_3 w_i^a = T\sqrt{2} \left(-u_i^a \sin(\vec{k} \cdot \vec{r}) + v_i^a \cos(\vec{k} \cdot \vec{r}) \right). \quad (5)$$

Choosing the temporal gauge, $A_0^a = 0$, the electric field components are simply the time derivatives of the vector potential

$$\begin{aligned} E_1^1 &= \dot{x} + \dot{w}_1^1, & E_2^1 &= \dot{w}_2^1, & E_3^1 &= 0, \\ E_1^2 &= \dot{w}_1^2, & E_2^2 &= \dot{y} + \dot{w}_2^2, & E_3^2 &= 0, \\ E_1^3 &= \dot{w}_1^3, & E_2^3 &= \dot{w}_2^3, & E_3^3 &= \dot{z}. \end{aligned} \quad (6)$$

The nonabelian magnetic field has the following components

$$\begin{aligned} B_1^1 &= -\partial_3 w_2^1 + gz(y + w_2^2), \\ B_2^1 &= \partial_3 w_1^1 - gzw_1^2, \\ B_3^1 &= g(w_1^2 w_2^3 - w_2^2 w_1^3) - gyw_1^3, \\ B_1^2 &= -\partial_3 w_2^2 - gzw_2^1, \\ B_2^2 &= \partial_3 w_1^2 + gz(x + w_1^1), \\ B_3^2 &= g(w_1^3 w_2^1 - w_2^3 w_1^1) - gxw_2^3, \\ B_1^3 &= -\partial_3 w_2^3, \\ B_2^3 &= \partial_3 w_1^3, \\ B_3^3 &= g(w_1^1 w_2^2 - w_2^1 w_1^2) + g(w_1^1 y + w_2^2 x) + gxy. \end{aligned} \quad (7)$$

In order to obtain the effective Hamiltonian for the soft modes, which are taken to be constant throughout space, we perform a spatial average over the relevant terms in the

Hamiltonian. For the quadratic and quartic expressions of the hard amplitudes w_i^a we use the following relations

$$\begin{aligned}
\overline{w_i^a w_j^b} &= u_i^a u_j^b + v_i^a v_j^b, \\
\overline{w_i^a \partial_3 w_j^b} &= T \left(u_i^a v_j^b - v_i^a u_j^b \right), \\
\overline{\partial_3 w_i^a \partial_3 w_j^b} &= T^2 \left(u_i^a u_j^b + v_i^a v_j^b \right), \\
\overline{w_i^a w_j^b w_i^a w_j^b} &= \frac{3}{2} \left(u_i^a u_i^a u_j^b u_j^b + v_i^a v_i^a v_j^b v_j^b + u_i^a v_i^a u_j^b v_j^b + u_i^a u_i^a v_j^b v_j^b \right).
\end{aligned} \tag{8}$$

The spatially averaged Hamiltonian,

$$\overline{H} = \frac{1}{2} \overline{E_i^a E_i^a} + \frac{1}{2} \overline{B_i^a B_i^a} \equiv H_T + H_S + H_I \tag{9}$$

contains three parts: H_T , describing the hard thermal modes interacting among themselves; H_S , describing the soft modes in isolation; and H_I denoting the interactions between hard and soft modes. According to the usual picture of a gluon plasma we treat the hard modes perturbatively, neglecting their mutual interactions. This yields:

$$H_T \longrightarrow H_0 = \frac{1}{2} \sum_{i=1}^2 \sum_{a=1}^3 \left[(\dot{u}_i^a \dot{u}_i^a + \dot{v}_i^a \dot{v}_i^a) + T^2 (u_i^a u_i^a + v_i^a v_i^a) \right], \tag{10}$$

in the leading order approximation.

The part of the Hamiltonian which contains soft modes must be treated non-perturbatively, so we keep the quartic anharmonic terms here. We get

$$H_S = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{g^2}{2} (x^2 y^2 + y^2 z^2 + z^2 x^2) \tag{11}$$

for the soft mode part and

$$H_I = \frac{1}{2} (M_1^1 x^2 + M_2^2 y^2 + (M_1^2 + M_2^1) xy + M_3^3 z^2) + Q^3 z \tag{12}$$

for the interaction between soft and hard thermal modes. Here

$$\begin{aligned}
M_1^1 &= g^2 (u_2^3 u_2^3 + v_2^3 v_2^3 + u_2^2 u_2^2 + v_2^2 v_2^2), \\
M_2^2 &= g^2 (u_1^3 u_1^3 + v_1^3 v_1^3 + u_1^1 u_1^1 + v_1^1 v_1^1),
\end{aligned}$$

$$\begin{aligned}
M_1^2 &= M_2^1 = g^2 \left(2u_1^1 u_2^2 + 2v_1^1 v_2^2 - u_2^1 u_1^2 - v_2^1 v_1^2 \right), \\
M_3^3 &= g^2 \left(u_1^1 u_1^1 + v_1^1 v_1^1 + u_2^1 u_2^1 + v_2^1 v_2^1 + u_1^2 u_1^2 + v_1^2 v_1^2 + u_2^2 u_2^2 + v_2^2 v_2^2 \right), \\
Q^3 &= 2gT \left(u_1^1 v_1^2 - u_2^2 v_2^1 + u_2^1 v_2^2 - u_1^2 v_1^1 \right)
\end{aligned} \tag{13}$$

are the components of the effective mass square matrix for the soft modes and Q^3 is the fluctuating color charge of the hard modes, which acts as a source for the soft modes.

The Hamiltonian H_0 (10) describes independent oscillations with a frequency of $\omega = |\vec{k}| = T$. For the dynamical evolution of the soft modes x, y, z this is a very fast motion; we can therefore average over many time periods in order to arrive at an effective soft mode Hamiltonian \overline{H}_I for the interaction. The ensemble averaging over the relative phases and slowly fluctuating amplitudes of the hard thermal components, however, cannot be done at this point, because these quantities change on the timescale $t_3 \sim 1/g^2 T$. Consider the hard thermal eigenmodes of the Hamiltonian (10):

$$\begin{aligned}
u_i^a(t) &= U_i^a \cos(Tt + \varphi_i^a), \\
v_i^a(t) &= V_i^a \cos(Tt + \psi_i^a).
\end{aligned} \tag{14}$$

The initial amplitudes U_i^a, V_i^a and phases φ_i^a, ψ_i^a ($i = 1, 2, a = 1, 2, 3$) determine the quantities \bar{M}_i^a and \bar{Q}^a which in turn influence the evolution of the soft modes x, y, z . In order to obtain the correct scaling $\bar{M}_i^a \sim g^2 T^2$ of the elements of the mass matrix (13) we take the amplitudes of the fast oscillations to be $\mathcal{O}(T)$. As seen from (10) this is consistent with an energy density $\bar{H}_0 \sim T^4$ in the hard modes, as expected for the thermal bath.

If, for the sake of simplicity, we assume all amplitudes equal to each other, so $U_i^a = V_i^a \equiv U \sim T$, we obtain $\bar{M}_1^1 = \bar{M}_2^2 = m^2, \bar{M}_3^3 = 2m^2$ with

$$m^2 = 2g^2 U^2 \sim g^2 T^2. \tag{15}$$

These effective mass terms in the averaged Hamiltonian \bar{H}_I are the analogue of the plasmon mass term in the full thermal gauge theory. For small amplitudes of the soft modes x, y, z they dominate over the anharmonic terms in (11) and apparently cause the soft modes to oscillate harmonically with a frequency $\omega_S \sim gT$.

However, the presence of non-diagonal elements M_1^2 in the mass square matrix can destabilize these oscillations. The time averaged off-diagonal term $\overline{M_1^2}$, as well as the source term $\overline{Q^3}$, depend on the phases of the hard thermal oscillations (14):

$$\overline{M_1^2} = \frac{m^2}{2} \left(\cos(\varphi_1^1 - \varphi_2^2) + \cos(\psi_1^1 - \psi_2^2) - \frac{1}{2} \cos(\varphi_2^1 - \varphi_1^2) - \frac{1}{2} \cos(\psi_2^1 - \psi_1^2) \right), \quad (16)$$

and

$$\overline{Q^3} = \frac{m^2 T}{2g} \left(\cos(\varphi_1^1 - \psi_1^2) - \cos(\varphi_2^2 - \psi_2^1) + \cos(\varphi_2^1 - \psi_2^2) - \cos(\varphi_1^2 - \psi_1^1) \right). \quad (17)$$

Because of their dependence on the relative phases of the fast modes $\overline{M_1^2}$ and $\overline{Q^3}$ would vanish if a complete ensemble average over the fast modes were performed. However, as pointed out at the beginning of our discussion, the phases and amplitudes of these modes change more slowly (on timescale t_3) than the soft modes oscillate (on timescale t_2). Hence, both $\overline{M_1^2}$ and $\overline{Q^3}$ must be considered as secular quantities in the averaged (over timescale t_1) Hamiltonian $H_S + \overline{H_I}$ governing the evolution of the soft modes.

Far from the “average” case, $\overline{M_1^2} = \overline{Q^3} = 0$, there is the extreme possibility of $\overline{M_1^2} = 3m^2/2$ and $\overline{Q^3} = m^2 T/2g \sim gT^3$ obtained for $\varphi_1^1 = \varphi_2^2 = \psi_1^2 = \psi_2^1 \pm \pi$, $\psi_1^1 = \psi_2^2 = \varphi_2^1 = \varphi_1^2 \pm \pi$. As we will see below, in this case the mass square matrix of the coupled x and y modes has negative eigenvalues giving rise to an *exponentially growing* solution of the linearized equations of motion for small amplitudes x and y .

We now consider small amplitude oscillations and neglect terms nonlinear in x or y . The z -coordinate then decouples from the $x - y$ motion, as seen from (13). It is most convenient to cast the resulting equation into matrix form

$$\begin{pmatrix} \frac{d^2}{dt^2} + \overline{M_1^2} & \overline{M_1^2} \\ \overline{M_1^2} & \frac{d^2}{dt^2} + \overline{M_2^2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (18)$$

where we remind the reader that the “bar” denotes the time averaged quantities for fixed phase φ_i^a, ψ_i^a and amplitudes U_i^a, V_i^a . Since the average over the ensemble of these initial conditions must be performed after the full solution is obtained, we proceed with solving (18) by finding the eigenvalues and eigenvectors of the matrix operator on the left hand side.

Seeking the solution with $x, y \propto e^{i\omega t}$, we find the eigenvalue equation

$$\omega^4 - \omega^2 (\overline{M_1^1} + \overline{M_2^2}) + (\overline{M_1^1} \overline{M_2^2} - \overline{M_1^2} \overline{M_2^1}) = 0. \quad (19)$$

Using the previous result $\overline{M_1^1} = \overline{M_2^2} = m^2$ and $\overline{M_1^2} = \overline{M_2^1} = \gamma m^2$ with $|\gamma| \leq 3/2$, we find

$$\omega_{\pm}^2 = m^2(1 \pm |\gamma|). \quad (20)$$

Clearly, for $|\gamma| > 1$ one of the eigenvalues becomes unstable with an imaginary frequency.

With the help of the eigenvalues ω_{\pm} and eigenvectors (x_{\pm}, y_{\pm}) the general solution of the time-averaged equation (13) can be written in the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \Delta(t) \begin{pmatrix} \dot{x}(0) \\ \dot{y}(0) \end{pmatrix} + \dot{\Delta}(t) \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}, \quad (21)$$

where

$$\Delta(t) = \sum_{i=\pm} \frac{\sin \omega_i t}{\omega_i} \begin{pmatrix} c_i^2 & c_i d_i \\ c_i d_i & d_i^2 \end{pmatrix}. \quad (22)$$

where (c_i, d_i) are the eigenvectors corresponding to the eigenvalues ω_i^2 . It is not difficult to show that the product $c_i d_i$ vanishes when it is averaged over the phases φ_i^a and ψ_i^a . As a result, the full thermal ensemble average gives rise to a color diagonal effective soft mode propagator $\langle \Delta(t) \rangle$, which contains an exponentially growing part originating from those regions of phase space where $\omega_-^2 < 0$, as is seen from (22).

Summarizing, we demonstrated in a simple model that the coupling of soft plasmon oscillations to hard thermal gluons can drive the soft field amplitude into an exponential growth and with that into a chaotic dynamical behavior. There are, of course, many possible few mode approximations to the Yang-Mills field theory. The example we discussed above was generic in the sense that it showed both dynamical mass generation and the development of chaotic instability due to off-diagonal parts of the self-energy, as well as source term coupling to the soft modes. One may expect that by this mechanism the dynamics of long wavelength modes in a hot gluon plasma remains chaotic despite the presence of a dynamical plasmon mass.

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